Solow Growth Model
Nurgul Ukueva

1 Preliminaries

In macroeconomics we study a dynamic economy. Each variable (output Y, labor L, capital K etc.) evolves over time, is dynamic. To incorporate these dynamics we will consider variables as functions of time: $Y(t)$, $K(t)$, $L(t)$ etc. (t stands for time). So, we are adding additional dimension to our model, dimension of time. In this growth part of the course we assume that time is continuous and that we observe variables at each instant of time.

Let variable $x(t)$ be a function of time t. Define $\dot{x}$ (called x *dot*) as an instantaneous change in x, that is $\dot{x}$ is the derivative of x with respect to time t:

$$\dot{x} \equiv \frac{dx(t)}{dt} \tag{1}$$

Define also, **growth rate of** x, g_x, as

$$g_x \equiv \frac{\dot{x}(t)}{x(t)} \tag{2}$$

If the growth rate of $x(t)$ is g_x and the growth rate of $y(t)$ is g_y and the variable $z = x^{\alpha}y^{\beta}$, then the growth rate of z is

$$g_z = \alpha g_x + \beta g_y$$

To see from: $z(t) = x(t)^{\alpha}y(t)^{\beta}$ take natural logarithms and then derivatives with respect to time from both sides:

$$\ln(z(t)) = \ln(x(t)^{\alpha}y(t)^{\beta}) = \alpha \ln(x(t)) + \beta \ln(y(t))$$

$$\frac{d\ln(z(t))}{dt} = \frac{d\left(\alpha \ln(x(t))\right)}{dt} + \beta\frac{d\left(\ln(y(t))\right)}{dt}$$

$$\frac{\dot{z}(t)}{z(t)} = \alpha\frac{\dot{x}(t)}{x(t)} + \beta\frac{\dot{y}(t)}{y(t)}$$

2 Solow model

Output Y is produced using labor L and capital K by the production function F:

$$Y = F(K, L) \tag{3}$$

Assumptions on the production function F:

- *Essentiality*: An input is essential if strictly positive amount of this input is needed to produce positive output. We assume that both capital and labor are essential for production. So, if $K = 0$ then $Y = F(0, L) = 0$. And if $L = 0$ then $Y = F(K, 0) = 0$
- *Positive Marginal products of labor and capital (MPL and MPK)*: For all $K > 0$ and $L > 0$

$$\begin{aligned} MPL \equiv \frac{\partial F(K,L)}{\partial L} &> 0 \\ MPK \equiv \frac{\partial F(K,L)}{\partial K} &> 0 \end{aligned} \tag{4}$$

- *Decreasing Marginal products of labor and capital*:

  For all $K > 0$ and $L > 0$

$$\begin{aligned} \frac{\partial^2 F(K,L)}{\partial L^2} &< 0 \\ \frac{\partial^2 F(K,L)}{\partial K^2} &< 0 \end{aligned} \tag{5}$$

- *Inada conditions:* Conditions on the marginal products of inputs (L, K) as these inputs approach 0 and ∞

$$\lim_{K \to 0}\left(\frac{\partial F(K,L)}{\partial K}\right) = \lim_{L \to 0}\left(\frac{\partial F(K,L)}{\partial L}\right) = \infty \tag{6}$$

$$\lim_{K\to\infty}\left(\frac{\partial F(K,L)}{\partial K}\right)=\lim_{L\to\infty}\left(\frac{\partial F(K,L)}{\partial L}\right)=0 \tag{7}$$

- *Constant returns to scale (CRS) with respect to labor and capital*: If we multiply both labor and capital by the same positive constant n, we get n times the amount of output:

$$F(nK,nL)=nF(K,L) \tag{8}$$

Cobb-Douglas production function of the form:

$$Y=K^{\alpha}L^{1-\alpha} \tag{9}$$

where $\alpha<1$ satisfies the above assumptions. In what follows we will work with the Cobb-Douglas production function. Note that all the qualitative results of the model hold true for any production function satisfying the above assumptions. We will work with a specific function, the Cobb Douglas function, for simplicity.

We will consider variables in per worker terms (or per capita terms, since we assume that labor input and population is the same in our model). Small case letters denote per capita terms: $y\equiv Y/L$ is output per capita, $k\equiv K/L$ is capital per capita. From our production function function because of its CRS property output per capita can be expressed as a function of capital per capita:

$$y\equiv\frac{Y}{L}=\frac{K^{\alpha}L^{1-\alpha}}{L}=K^{\alpha}L^{-\alpha}=k^{\alpha} \tag{10}$$

We assume that the labor supply equals population and population is growing exogenously (not determined within the model but given as a parameter) over time at a constant rate n. That is population is given as the following function of time:

$$L(t)=L_0e^{nt} \tag{11}$$

where L_0 is the initial population, i. e. population at time $t=0$ and n is the growth rate of population. This growth rate of population is *constant and exogenous*

What people in this economy do over time is that they use initial level of capital stock K_0, which is given exogenously, produce output $Y_0 = K_0^\alpha L_0^{1-\alpha}$. Then they consume some part of this output and invest the rest into capital. We will assume in the Solow model that savings are equal to investment, and people save and invest sY of the their output, and consume the rest of the output, $(1-s)Y$. The savings rate s in this model is constant (i.e. does not change over time) and exogenous. Next period the capital stock of the economy is going to be equal to the original capital stock K_0 plus investment, sY, minus depreciation δK. We assume, that part of the capital stock δK depreciates and that the depreciation rate δ is also constant and exogenous. Then the economy in the next period uses the new level of capital stock, produces output, out of this output saves and invests part sY and consumes the rest, capital stock again changes due to investment and depreciation and this continues on and on, every period.

If time is continuous and the change in time is instantaneous as in our model then the change in the capital stock is given by:

$$\dot{K} = sY - \delta K \tag{12}$$

This equation is called the Law of motion of capital stock. It means that the instantaneous change in the capital stock, the $\dot{K}$ term, is due to investment and depreciation. So, capital stock increases as we invest more and decreases because part of it weirs out and becomes useless due to depreciation. The equation (12) is a differential equation.

We can divide both sides of the equation above to get growth rate of capital stock:

$$\frac{\dot{K}}{K} = \frac{sY}{K} - \delta \tag{13}$$

So I have 2 equations that govern the economy: the production function and the equation for the growth rate of the capital stock:

$$Y = K^\alpha L^{1-\alpha} \tag{14}$$

$$\frac{\dot{K}}{K} = \frac{sY}{K} - \delta \tag{15}$$

In order to analyze this economy, I transform the 2 equations above in per capita terms. (Notice the mathematical reason I am doing this transformation is because my population L is growing exogenously, so I want to divide all variables by population to get rid of this exogenous growth factor and try to form, if possible, stable variables). We have already done the transformation of the production function (this transformation is possible due to CRS property of the production function):

$$y \equiv \frac{Y}{L} = \frac{K^{\alpha}L^{1-\alpha}}{L} = K^{\alpha}L^{-\alpha} = k^{\alpha} \tag{16}$$

Let's look at the equation for the growth rate of capital stock. First divide and multiply sY/K term by L to get per capita terms on the right hand side:

$$\frac{\dot{K}}{K} = \frac{sY/L}{K/L} - \delta = \frac{sy}{k} - \delta \tag{17}$$

Now on the left hand side, realize that by definition:

$$K \equiv kL$$

That means, that using our results in the Preliminaries section of this note:

$$\frac{\dot{K}}{K} = \frac{\dot{k}}{k} + \frac{\dot{L}}{L} = \frac{\dot{k}}{k} + n$$

The growth rate of the total capital stock equals the growth rate of the per capita capital stock plus growth rate of population.

Substituting this into equation (17) above gives:

$$\frac{\dot{k}}{k} + n = \frac{sy}{k} - \delta$$

Or:

$$\frac{\dot{k}}{k} = \frac{sy}{k} - (\delta + n)$$

Notice that $y/k = (yL)/(kL) = Y/K$ is the average product of capital, and it is important in our differential equation. We have not yet substituted for the production function, this is equation is true for any production function. Let's substitute for y, from equation (16) we know that $y = k^\alpha$, therefore:

$$\frac{\dot{k}}{k} = \frac{sk^\alpha}{k} - (\delta + n)$$

Therefore the final equation we have is:

$$\frac{\dot{k}}{k} = sk^{\alpha-1} - (\delta + n) \tag{18}$$

Long-run or Steady State

In the long-run the economy reaches its steady state. The capital stock per capita, k^{ss} will be constant, growth rate of the capital stock per capita in the steady state $\dot{k^{ss}}/k^{ss}$ will be 0.

So for the steady state we set $\dot{k^{ss}}/k^{ss} = 0$, and equation (18) becomes:

$$\frac{\dot{k}}{k} = sk^{\alpha-1} - (\delta + n) = 0 \tag{19}$$

We can solve this equation for the capital stock per capita in the steady state:

$$k^{ss} = \left(\frac{s}{n + \delta}\right)^{\frac{1}{1-\alpha}} \tag{20}$$

Since we know that $y = k^\alpha$, output per capita in the steady state is:

$$y^{ss} = (k^{ss})^\alpha = \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{21}$$

Thus, in this basic Solow model output per capita and capital per capita in the steady state are constant, the growth rates of both of them in the long-run, steady state, are 0.

As for their levels in this model, the factors that affect the long-run **levels** of output per capita and capital per capita are savings rate s, depreciation rate δ and growth rate of population n.

3 Solow model with exogenous technological growth

As you can see, in the basic Solow model with no change in technology, output per capita and capital stock per capita are constant in the long-run. So, we have an economy that has constant income or GDP per capita in the long-run. This is a counterfactual result. We know, for example, that for more than a century the world's economy, economies of OECD countries or US economy have been steadily growing.

In order to take this long-run growth in the real world into account we modify the basic Solow model, and include technology that is growing exogenously. So, we add labor-augmenting technology (technology entered as a multiple of labor in the production function) into our production function.

$$Y = K^\alpha (AL)^{1-\alpha} \tag{22}$$

This technology A is growing over time at a constant and exogenous rate g_a. Before we go any further with the analysis of this model, you might guess, that since there is a technological progress, and technology is steadily improving over time, our model will generate steady state growth of output per capita and capital stock per capita. So, in contrast to the basic Solow model with no technological progress in the previous section, in this model the growth rate of y^{ss} and k^{ss} will be positive not 0.

Again we assume that the population is growing at an exogenous rate n. The economy saves constant fraction of its output sY, savings equal investment. Savings rate s is constant and exogenous. Capital stock again depreciates at a rate δ. So, we have the usual law of motion of total capital stock:

$$\dot{K} = sY - \delta K \tag{23}$$

Or in growth terms, dividing both sides by K

$$\frac{\dot{K}}{K} = \frac{sY}{K} - \delta \tag{24}$$

Again, we have 2 equations that govern the economy: the modified production function with growing technology and the equation for the growth rate of the capital stock:

$$Y = K^{\alpha}(AL)^{1-\alpha} \tag{25}$$

$$\frac{\dot{K}}{K} = \frac{sY}{K} - \delta \tag{26}$$

In order to analyze this economy, for mathematical reasons I have to transform the equations into so called per effective labor terms. Because now, not only my population L is growing exogenously, but also my technology is growing exogenously. So I want to divide all variables by population **and technology** to get rid of this exogenous growth factors and try to form stable variables, which are now per effective labor variables.

Therefore, we define *capital per effective labor* and *output per effective labor* as follows:

$$\tilde{k} \equiv \frac{K}{AL}$$

$$\tilde{y} \equiv \frac{Y}{AL}$$

To transform the production function equation:

$$\tilde{y} \equiv \frac{Y}{AL} = \frac{K^{\alpha}(AL)^{1-\alpha}}{AL} = \left(\frac{K}{AL}\right)^{\alpha} = \tilde{k}^{\alpha} \tag{27}$$

And to transform the equation for the growth rate of capital stock, divide and multiply sY/K term by AL

$$\frac{\dot{K}}{K} = \frac{sY}{K} - \delta = \frac{sY/AL}{K/AL} - \delta = \frac{s\tilde{y}}{\tilde{k}} - \delta \tag{28}$$

Now again on the left hand side, realize that by definition:

$$K \equiv \tilde{k}AL$$

That means, that:

$$\frac{\dot{K}}{K} = \frac{\dot{\tilde{k}}}{\tilde{k}} + \frac{\dot{L}}{L} + \frac{\dot{A}}{A} = \frac{\dot{\tilde{k}}}{\tilde{k}} + n + g_a$$

The growth rate of the total capital stock equals the growth rate of the capital stock per effective labor plus growth rate of population and plus the growth rate of technology.

Substituting this into equation (28) gives:

$$\frac{\dot{\tilde{k}}}{\tilde{k}} + n + g_a = \frac{s\tilde{y}}{\tilde{k}} - \delta$$

Or:

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{s\tilde{y}}{\tilde{k}} - (\delta + n + g_a)$$

Finally, to get an equation only in terms of $\tilde{k}$, substitute from our transformed production function $\tilde{y} = \tilde{k}^{\alpha}$:

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = s\tilde{k}^{\alpha-1} - (\delta + n + g_a) \tag{29}$$

Long-run or Steady State

To analyze the long-run for this economy with growing technology we have to look at per effective labor variables. These are the variables, that will not grow, will be constant, in the steady state. So, capital per effective labor, $\tilde{k}^{ss}$ will be constant.

So for the steady state we set $\dot{\tilde{k}}^{ss}/\tilde{k}^{ss} = 0$, and equation (29) becomes:

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = s\tilde{k}^{\alpha-1} - (\delta + n + g_a) = 0 \tag{30}$$

We can solve for the steady state level of capital per effective labor:

$$\tilde{k}^{ss} = \left(\frac{s}{n + \delta + g_a}\right)^{\frac{1}{1-\alpha}} \tag{31}$$

Since we know that $\tilde{y} = \tilde{k}^{\alpha}$, output per effective labor in the steady state is:

$$\tilde{y}^{ss} = \left(\tilde{k}^{ss}\right)^{\alpha} = \left(\frac{s}{n + \delta + g_a}\right)^{\frac{\alpha}{1-\alpha}} \tag{32}$$

Let me stress again that with growing technology the stable, constant variables are per effective labor variables. But we are as economists interested in per capita variables. How do we find per capita variables? Since $\tilde{k} \equiv K/(AL)$ and $\tilde{y} \equiv Y/(AL)$, by definition capital stock per capita and output per capita are:

$$k = \frac{K}{AL}A = \tilde{k}A$$

$$y = \frac{Y}{AL}A = \tilde{y}A$$

And in the steady state (or for this growing economy we say along the balanced growth path) capital and output per capita will be growing at a constant rate g_a, the growth rate of technology.

Since the $\tilde{k}^{ss}$ and $\tilde{y}^{ss}$ are constant but technology is exogenously growing we can write the long-run capital and output per capita as:

$$k(t)^{ss} = \tilde{k}^{ss} A(t) = \left(\frac{s}{n + \delta + g_a}\right)^{\frac{1}{1-\alpha}} A(t)$$

$$y(t)^{ss} = \tilde{y}^{ss} A(t) = \left(\frac{s}{n + \delta + g_a}\right)^{\frac{\alpha}{1-\alpha}} A(t)$$

Notice I purposefully put time variable in brackets, to stress that these per capita variables are time dependent even in the long-run. In contrast to our basic Solow model they are growing over time. These levels per capita are changing, steadily growing over time, even if the exogenous parameters of the model such as savings, depreciation, growth of population, growth of technology are not altered.

The long-run growth rate of capital and output per capita in this model is affected only by the growth rate of technology g_a.

Growth rate of output per capita in the steady state:

We have solved the steady state equation for per effective labor variables, shown that we have a unique steady state, where the growth rate of steady state output per effective labor and capital per effective labor is 0. Then by definition we got the per capita variables as:

$$k = \frac{K}{AL} A = \tilde{k} A$$

$$y = \frac{Y}{AL} A = \tilde{y} A$$

which means:

$$\frac{\dot{k}}{k} = \frac{\dot{\tilde{k}}}{\tilde{k}} + \frac{\dot{A}}{A}$$

$$\frac{\dot{y}}{y} = \frac{\dot{\tilde{y}}}{\tilde{y}} + \frac{\dot{A}}{A}$$

In the steady state, since $\tilde{k}$ and $\tilde{y}$ are constant, it has to be that both y and k grow at rate of growth of A. So, we have already shown that both k and y grow at a rate g_a.

Let's now check if this is consistent with our production function:

Our production function is

$$Y = (AL)^{1-\alpha} K^{\alpha}$$

Per capita output y is

$$y \equiv \frac{Y}{L} = \frac{(AL)^{1-\alpha} K^{\alpha}}{L} = A^{1-\alpha} L^{-\alpha} K^{\alpha} = A^{1-\alpha} \left(\frac{K}{L}\right)^{\alpha} = A^{1-\alpha} k^{\alpha}$$

which means the growth rate of per capita output is:

$$\frac{\dot{y}}{y} = (1-\alpha)\frac{\dot{A}}{A} + \alpha\frac{\dot{k}}{k}$$

if all y and k and A grow at the same rate then:

$$g_a = (1-\alpha) g_a + \alpha g_a$$

$$g_a = g_a$$

That is the result we obtained about the growth rate in the steady state is consistent with our production function.

Note in this model $y = A^{1-\alpha}k^{\alpha}$, not k^{α} anymore (although $\tilde{y} = \tilde{k}^{\alpha}$). Since A is growing we cannot say that y is growing at a rate αg_k.

The conclusion is growth rate of y equals growth rate of k equals g_a